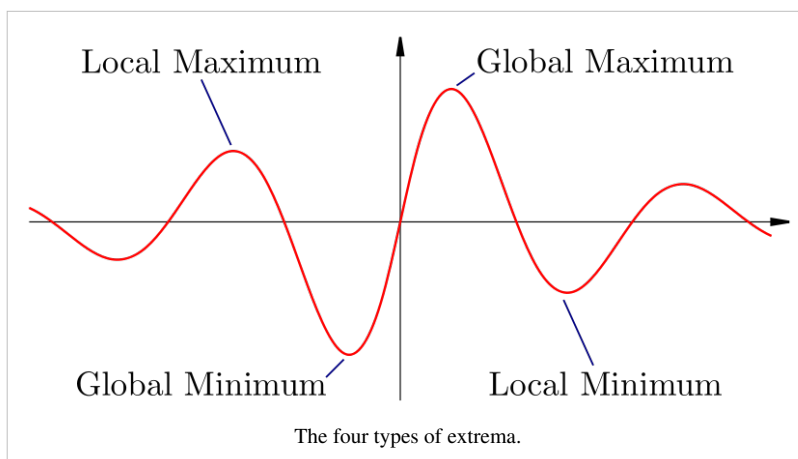


Calculus/Extrema and Points of Inflection

Maxima and **minima** are points where a function reaches a highest or lowest value, respectively. There are two kinds of **extrema** (a word meaning maximum *or* minimum): **global** and **local**, sometimes referred to as "absolute" and "relative", respectively. A global maximum is a point that takes the largest value on the entire range of the function, while a global minimum is the point that takes the smallest value on the range of the function. On

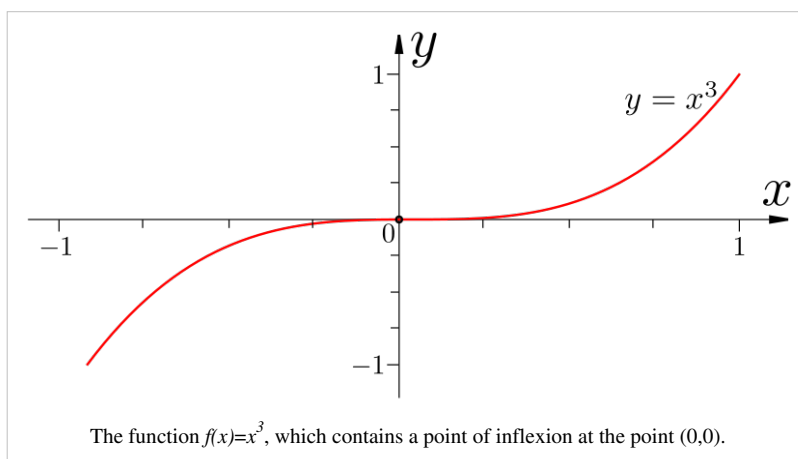


the other hand, local extrema are the largest or smallest values of the function in the immediate vicinity.

All extrema look like the crest of a hill or the bottom of a bowl on a graph of the function. A global extremum is always a local extremum too, because it is the largest or smallest value on the entire range of the function, and therefore also its vicinity. It is also possible to have a function with no extrema, global or local: $y=x$ is a simple example.

At any extremum, the slope of the graph is necessarily zero, as the graph must stop rising or falling at an extremum, and begin to head in the opposite direction. Because of this, extrema are also commonly called **stationary points** or **turning points**. Therefore, the first derivative of a function is equal to zero at extrema. If the graph has one or more of these stationary points, these may be found by setting the first derivative equal to zero and finding the roots of the resulting equation.

However, a slope of zero does not guarantee a maximum or minimum: there is a third class of stationary point called a point of inflection (also spelled point of inflection). Consider the function



$$f(x) = x^3.$$

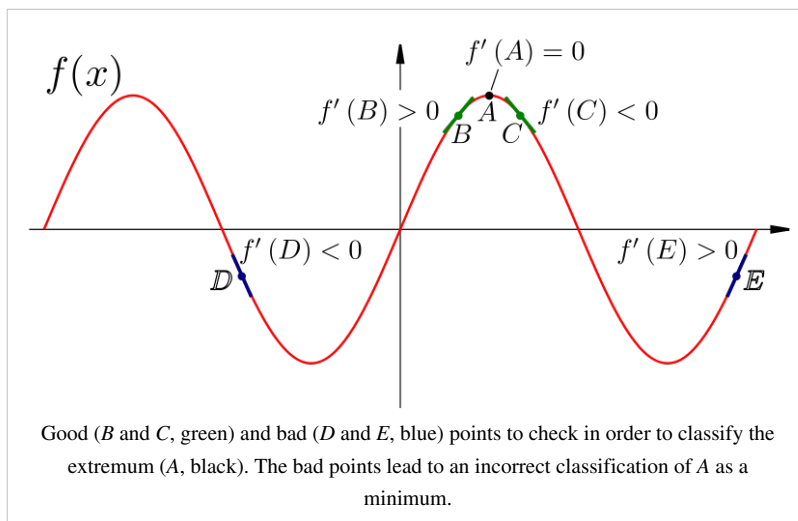
The derivative is

$$f'(x) = 3x^2$$

The slope at $x=0$ is 0. We have a slope of zero, but while this makes it a stationary point, this doesn't mean that it is a maximum or minimum. Looking at the graph of the function you will see that $x=0$ is neither, it's just a spot at which the function flattens out. True extrema require the a sign change in the first derivative. This makes sense - you have to rise (positive slope) to and fall (negative slope) from a maximum. In between rising and falling, on a smooth curve, there will be a point of zero slope - the maximum. A minimum would exhibit similar properties, just in

reverse.

This leads to a simple method to classify a stationary point - plug x values slightly left and right into the derivative of the function. If the results have opposite signs then it is a true maximum/minimum. You can also use these slopes to figure out if it is a maximum or a minimum: the left side slope will be positive for a maximum and negative for a minimum. However, you must exercise caution with this method, as, if you pick a point too far from the extremum, you could take it on the far side of *another* extremum and incorrectly classify the point.



The Extremum Test

A more rigorous method to classify a stationary point is called the **extremum test**, or **2nd Derivative Test**. As we mentioned before, the sign of the first derivative must change for a stationary point to be a true extremum. Now, the *second* derivative of the function tells us the rate of change of the first derivative. It therefore follows that if the second derivative is positive at the stationary point, then the gradient is increasing. The fact that it is a stationary point in the first place means that this can only be a minimum. Conversely, if the second derivative is negative at that point, then it is a maximum.

Now, if the second derivative is zero, we have a problem. It could be a point of inflexion, or it could still be an extremum. Examples of each of these cases are below - all have a second derivative equal to zero at the stationary point in question:

- $y = x^3$ has a point of inflexion at $x = 0$
- $y = x^4$ has a minimum at $x = 0$
- $y = -x^4$ has a maximum at $x = 0$

However, this is not an insoluble problem. What we must do is continue to differentiate until we get, at the $(n+1)$ th derivative, a non-zero result at the stationary point:

$$f'(x) = 0, f''(x) = 0, \dots, f^{(n)}(x) = 0, f^{(n+1)}(x) \neq 0$$

If n is odd, then the stationary point is a true extremum. If the $(n+1)$ th derivative is positive, it is a minimum; if the $(n+1)$ th derivative is negative, it is a maximum. If n is even, then the stationary point is a point of inflexion.

As an example, let us consider the function

$$f(x) = -x^4$$

We now differentiate until we get a non-zero result at the stationary point at $x=0$ (assume we have already found this point as usual):

$$f'(x) = -4x^3$$

$$f''(x) = -12x^2$$

$$f'''(x) = -24x$$

$$f^{(4)}(x) = -24$$

Therefore, $(n+1)$ is 4, so n is 3. This is odd, and the fourth derivative is negative, so we have a maximum. Note that none of the methods given can tell you if this is a global extremum or just a local one. To do this, you would have to set the function equal to the height of the extremum and look for other roots.

See "Optimization" for a common application of these principles.

Article Sources and Contributors

Calculus/Extrema and Points of Inflection *Source:* <http://en.wikibooks.org/w/index.php?oldid=1661739> *Contributors:* Mrwojo, 1 anonymous edits

Image Sources, Licenses and Contributors

Image:Maxima and Minima.svg *Source:* http://en.wikibooks.org/w/index.php?title=File:Maxima_and_Minima.svg *License:* Public Domain *Contributors:* User:Inductiveload

Image:X_cubed_(narrow).svg *Source:* [http://en.wikibooks.org/w/index.php?title=File:X_cubed_\(narrow\).svg](http://en.wikibooks.org/w/index.php?title=File:X_cubed_(narrow).svg) *License:* Public Domain *Contributors:* User:Inductiveload

image:Extremum Testing.svg *Source:* http://en.wikibooks.org/w/index.php?title=File:Extremum_Testing.svg *License:* Public Domain *Contributors:* User:Inductiveload

License

Creative Commons Attribution-Share Alike 3.0 Unported
<http://creativecommons.org/licenses/by-sa/3.0/>
